Find the Maclaurin Series (Taylor Series at x=0) for

$$f(x) = (1+x)^m \leftarrow \text{some constant}$$

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$$

$f^{(0)}(x) = (1+x)^m$
$f^{(1)}(x) = m(1+x)^{m-1}$
$f^{(2)}(x) = m(m-1)(1+x)^{m-2}$
$f^{(3)}(x) = m(m-1)(m-2)(1+x)^{m-3}$
$\vdots$
$f^{(k)}(x) = m(m-1)\cdots(m-[k-1])(1+x)^{m-k}$

$f^{(0)}(0) = (1+0)^m = 1^m = 1$
$f^{(1)}(0) = m(1+0)^{m} = m1^{m} = m$
$f^{(2)}(0) = m(m-1)\cancel{(1+0)^{m}} = m(m-1)$
$\vdots$
$f^{(k)}(0) = m(m-1)(m-2)\cdots\overset{(m-k+1)}{(m-[k-1])}$
$= \prod_{i=0}^{k-1}(m-i)$

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = \sum_{k=0}^{\infty} \frac{m(m-1)\cdots(m-k+1)}{k!} x^k$$

$$= \sum_{k=0}^{\infty} \frac{\prod_{i=0}^{k-1}(m-i)}{k!} x^k$$

$$= \sum_{k=0}^{\infty} \binom{m}{k} x^k$$

$$\binom{m}{k} = \frac{m(m-1)\cdots(m-k+1)}{k!} = \frac{\prod_{i=0}^{k-1}(m-i)}{k!}$$

(If m,k are integers $= \dfrac{m!}{k!(m-k)!}$)

A few notes...

$$\prod_{i=0}^{k-1}(m-i) \quad \text{for } k=0 \text{ is } \prod_{i=0}^{-1}(m-i)=1$$

OR

$$\sum_{k=0}^{\infty}\frac{m(m-1)\cdots(m-k+1)}{k!}x^k = 1+\sum_{k=1}^{\infty}\frac{m(m-1)\cdots(m-k+1)}{k!}x^k$$

I never actually showed that the Maclaurin Series converged or that it converged to the generating function...

- Use Ratio Test to show
$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right|<1 \quad \text{for } |x|<1 \ \checkmark$$

- Also, you can prove
$$\lim_{n\to\infty}|R_n(x)| = \lim_{n\to\infty}\left|\frac{f^{(k)}(c_n)}{(n+1)!}x^{n+1}\right| \leq 0$$
when $|x|<1$. So the Maclaurin Series does converge to $(1+x)^m$ for $|x|<1$

Bottom line:

$$(1+x)^m = \sum_{k=0}^{\infty}\frac{m(m-1)\cdots(m-k+1)}{k!}x^k$$

Examples:

Find the first 3 terms of the expansion of $(1+x)^{99}$.

$$(1+x)^{99} = \sum_{k=0}^{\infty} \frac{m(m-1)\cdots(m-k+1)}{k!} x^k$$

$$= \frac{1}{0!}x^0 + \frac{m}{1!}x^1 + \frac{m(m-1)}{2!}x^2 + \cdots$$

$$= 1 + mx + \frac{m(m-1)}{2}x^2 + \cdots$$

$$= 1 + 99x + \frac{99(98)}{2}x^2 + \cdots$$

$$= \boxed{1 + 99x + 99(49)x^2} + \cdots$$

Find a polynomial of degree 4 which approximates $(1+x)^{1/2} = \sqrt{1+x}$ for $|x|<1$

$$(1+x)^{1/2} = \sum_{k=0}^{\infty} \frac{\prod_{i=0}^{k-1}(m-i)}{k!} x^k$$

$$= \cancel{\frac{\prod_{i=0}^{-1}(m-i)}{0!} x^0} + \frac{\prod_{i=0}^{0}(m-i)}{1!} x^1 + \frac{\prod_{i=0}^{1}(m-i)}{2!} x^2$$

$$+ \frac{\prod_{i=0}^{2}(m-i)}{3!} x^3 + \frac{\prod_{i=0}^{3}(m-i)}{4!} x^4 + \cdots$$

$$= 1 + \frac{m}{1} x + \frac{m(m-1)}{2} x^2$$

$$+ \frac{m(m-1)(m-2)}{6} x^3 + \frac{m(m-1)(m-2)(m-3)}{24} x^4 + \cdots$$

$$= 1 + \tfrac{1}{2}x + \frac{\frac{1}{2}(-\frac{1}{2})}{2} x^2 + \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{\cancel{3}^{1}}{2})}{\cancel{6}_{2}} x^3$$

$$+ \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{\cancel{3}^{1}}{2})(-\frac{5}{2})}{\cancel{24}\,8} x^4 + \cdots$$

$$= \boxed{1 + \tfrac{1}{2}x - \tfrac{1}{8}x^2 + \tfrac{1}{16}x^3 - \tfrac{5}{128}x^4} + \cdots$$

Find a rational function to approximate

$\left(1+\frac{1}{x}\right)^{1/2}$.

$(1+x)^{1/2} \approx$

$$1 + \tfrac{1}{2}x - \tfrac{1}{8}x^2 + \tfrac{1}{16}x^3 - \tfrac{5}{128}x^4$$

$\left(1+\frac{1}{x}\right)^{1/2} \approx$

$$1 + \frac{1}{2x} - \frac{1}{8x^2} + \frac{1}{16x^3} - \frac{5}{128x^4}$$