$$\sum_{n=0}^{\infty} ar^n = \sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \quad \text{for } |r|<1$$

Geometric Series

8.7 Power Series

Idea: We want to be able to consider "infinite" polynomials

$$f(x) = 1 + x + x^2 + x^3 + x^4 + \cdots = \sum_{n=0}^{\infty} x^n$$

We can't plug in any number for x, necessarily...

$$f(1) = 1 + 1 + 1 + 1 + 1 + \cdots \to \infty$$

So 1 is not in the domain

$$f(-2) = 1 - 2 + 4 - 8 + 16 - 32 + \cdots = \sum_{n=0}^{\infty} (-2)^n$$

Diverges b/c geometric series diverge for $|r| = |-2| > 1$.

But other numbers do make sense:

$$f(1/2) = 1 + 1/2 + 1/4 + 1/8 + \cdots = \sum_{n=0}^{\infty} (1/2)^n = \frac{1}{1 - 1/2} = 2$$

$$f(-1/3) = 1 - 1/3 + 1/9 - 1/27 + \cdots = \sum_{n=0}^{\infty} (-1/3)^n = \frac{1}{1 + 1/3} = 3/4$$

Two questions:
(1) How do we find the domain of a "power series"?
(2) How do we express a "power series" without a summation symbol?

(1) How do we find the domain of a "power series"?

$$f(x) = \sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} (1)(x)^n \leftarrow \text{Geometric Series with } a=1,\ r=x$$

When does this series **converge**? (These are the values that we'll be able to plug into f(x) and get a real number as a value.)

Geo. Series converges for $\cancel{|r|<1}$ $|x|<1$

Our "~~domain~~ interval of convergence" is:

$|x|<1$ $\boxed{\text{OR}}$ $-1<x<1$ $\boxed{\text{OR}}$ $(-1,1)$

A power series about x=a is:

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 \cancel{(x-a)^0} + c_1 (x-a)^{\cancel{1}} + c_2 (x-a)^2 + \cdots$$

$$= c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

We say the **center** of this power series is a, and the **coefficients** of this power series are $c_0$, $c_1$, $c_2$, etc.

Ex 1

$\sum_{n=0}^{\infty} x^n$ is a power series

with $a=0$ and $c_n = 1$

How do we find the domain/**interval of convergence** of a random power series?

Answer: Ratio Test, typically.

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1}$$

$$\Big(= x \sum_{n=0}^{\infty} \underbrace{\frac{(-1)^n}{n+1}}_{c_n} \underbrace{x^n}_{(x-0)^n}\Big) \leftarrow \text{Power Series}$$

$a=0$

$c_n = \frac{(-1)^n}{n+1}$

Check

$$\lim_{n\to\infty} \left| \frac{\cancel{(-1)^n} \frac{x^{n+1}}{n+1}}{\cancel{(-1)^{n-1}} \frac{x^n}{n}} \right| = \lim_{n\to\infty} \left| \frac{x^{\cancel{n+1}}}{n+1} \cdot \frac{n}{\cancel{x^n}} \right|$$

$$= \lim_{n\to\infty} |x| \left| \frac{n}{n+1} \right|$$

$$= |x| \boxed{\cancel{\lim_{n\to\infty} \left| \frac{n}{n+1} \right|}} = |x|$$

Our series converges if that limit is less than one, so...

$|x| < 1$ is in the domain of our series

b/c $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$ is absolutely convergent

for $|x| < 1$

Steps to determine the interval of convergence of a series:

1. Use Ratio Test (or Root Test) to find the interval where the series converges absolutely.

   Ex: $|x-a| < R$

2. If the interval of convergence is finite, test $|x-a| = R$ using some other convergence test

3. If $|x-a| > R$ then it diverges.

New Ratio Test:

If $\sum_{n=0}^{\infty} a_n$ is a series and

① $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$ it converges

② $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$ it diverges