Definition of Sequences, Sequence Limits, and Convergence/Divergence

A sequence is a function from the positive integers to the real numbers. Equivalently, it can be thought of as an infinite list of numbers. The sequence a_n is said to have a limit $\lim\limits_{n\to\infty} a_n$ if the list of numbers becomes arbitrarily close to a single number further down the list.

A sequence **converges** if it has a finite limit. A sequence **diverges** if it does not have a finite limit. It diverges to $\pm\infty$ if the sequence is a list of numbers growing to be arbitrarily positive or negative numbers, respectively.

Relationship Between Function and Sequence Limits

Suppose that $f(x)$ is a function and a_n is a sequence such that $f(i) = a_i$ for all integers i. If $\lim\limits_{x\to\infty} f(x)$ exists, then $\lim\limits_{n\to\infty} a_n = \lim\limits_{x\to\infty} f(x)$.

How to use this:

- A valid summary of all of this is to say that the limit of a sequence can be treated the same way as the limit of a similar function.
- This allows us to use all of our rules for the limits of functions. Even L'Hopital's Rule can be applied.
- However, if we're dealing with sequences involving factorials, recursion, or other concepts that don't make sense for functions, we have to use our intuition to determine how the sequence changes. (The more precise method for finding such limits requires using a more technical definition of limit.)
- Often listing the first few terms of a sequence helps us determine the eventual limit of a sequence.

Examples:

1. Does $a_n = \dfrac{n^2}{n^2+n}$ converge or diverge?
   Answer:
   $$\lim_{n\to\infty} \frac{n^2}{n^2+n} = \lim_{n\to\infty} \frac{n^2}{n^2} = 1$$
   Thus a_n **converges** to 1.
2. Does $a_n = \cos(\pi n)$ converge or diverge?
   Answer:
   $a_n = \langle \cos(\pi), \cos(2\pi), \cos(3\pi), \cos(4\pi), ...\rangle = \langle -1, 1, -1, 1, ...\rangle$, so a_n **diverges**.
   Note
3. Does $a_n = \sin(\pi n)$ converge or diverge?
   Answer:
   $a_n = \langle \sin(\pi), \sin(2\pi), \sin(3\pi), \sin(4\pi), ...\rangle = \langle 0, 0, 0, 0, ...\rangle \to 0$, so a_n **converges** to 0.

4. Does $a_n = \dfrac{\ln(n^2)}{3n}$ converge or diverge?

   Answer:

$$\lim_{n\to\infty} \frac{\ln(n^2)}{3n} = \lim_{n\to\infty} \frac{2\ln n}{3n} =^{LH} \lim_{n\to\infty} \frac{2/n}{3} = \frac{0}{3}$$

   Thus a_n **converges** to 0.

5. Does $a_n = \dfrac{\cos(\pi n)}{n!} + \dfrac{2n}{n+1}$ converge or diverge?

   Answer:

   Note first that $\dfrac{\cos(\pi n)}{n!} = \dfrac{(-1)^n}{n!} = \left\langle -\frac{1}{1}, \frac{1}{2}, -\frac{1}{6}, \frac{1}{24}, ... \right\rangle \to 0$. So:

$$\lim_{n\to\infty} \frac{\cos(\pi n)}{n!} + \frac{2n}{n+1} = \lim_{n\to\infty} \frac{\cos(\pi n)}{n!} + \lim_{n\to\infty} \frac{2n}{n+1} = 0 + 2 = 2$$

   Thus a_n **converges** to 2.

Upper Bound Test for Sequences

If a_n has an upper bound (that is, a number N such that $a_i \leq N$ for any integer i) and is nondecreasing, a_n converges. If a_n has no upper bound, a_n diverges.

How to use this:

- If you can know that a sequence never decreases (that is, $a_n \leq a_{n+1}$ always) and you can show a number that is bigger than every term of the sequence, you know the sequence converges.
- If you can show that a sequence always has a term larger than any number you can think of, then it diverges.

Examples:

1. Does $a_n = \dfrac{n}{n+1}$ converge or diverge?

   Answer:

   - $a_n = \left\langle \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, ... \right\rangle$ never decreases
   - $\dfrac{n}{n+1} < 382$ for any n.

   Thus by the upper bound test, a_n **converges**. (More work can show it converges to 1.)

2. Does $a_n = (-10)^n$ converge or diverge?

   Answer:

   For even powers, a_n grows to arbitrarily large positive numbers. Thus it has no upper bound and **diverges**. (Note it does not diverge to ∞ as all the terms aren't becoming arbitrarily large positive numbers.)

Definition of a Partial Sum Sequence and Series

If a_n is a sequence, then $s_n = \sum_{i=1}^{n} a_i$ is its partial sum sequence. A series $\sum_{i=1}^{\infty} a_i$ is the limit of the partial sum sequence of a_i. That is, $\sum_{i=1}^{\infty} a_i = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \sum_{i=1}^{n} a_i$. A series converges/diverges when the partial sum sequence converges/diverges.

Reindexing a Series

1. For any integers j, k, $\sum_{n=k}^{\infty} a_n = \sum_{n=k+j}^{\infty} a_{n-j}$.
2. $\sum_{n=1}^{\infty} a_n = a_1 + \sum_{n=2}^{\infty} a_n$ and $\sum_{n=2}^{\infty} a_n = \left(\sum_{n=1}^{\infty} a_n\right) - a_1$
3. If $\sum_{n=K}^{\infty} a_n$ converges or diverges for some integer K, then $\sum_{n=K}^{\infty} a_n$ converges or diverges (respectively) for ANY integer K, provided that a_n is defined for all integers greater than or equal to K.

How to use this:

1. You may adjust the starting index of a series as long as you do the opposite to the inside of the series.
2. You may also adjust the starting index of a series by taking out the terms you lose by moving the starting index forward, or subtracting away the terms you add by moving the starting index backward.
3. If it's easier to check the convergence/divergence of a series by changing the starting index, you're allowed to, as long as you don't cause a divide by zero error or similar.

Examples:

1. These are all equal:

$$\sum_{k=3}^{\infty} \frac{k}{2^{k+1}} \qquad \sum_{k=1}^{\infty} \frac{k+2}{2^{k+3}} \qquad \left(\sum_{k=1}^{\infty} \frac{k}{2^{k+1}}\right) - \frac{1}{2^2} - \frac{2}{2^3} \qquad \frac{3}{2^4} + \frac{4}{2^5} + \sum_{k=3}^{\infty} \frac{k+2}{2^{k+3}}$$

2. If you show $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, you've shown $\sum_{n=2}^{\infty} \frac{1}{n^2}, \sum_{n=7}^{\infty} \frac{1}{n^2}$ (etc.) converge. You haven't shown $\sum_{n=0}^{\infty} \frac{1}{n^2}$ converges, because that series isn't well-defined. (It would require you to plug zero into a denominator.) Also, note that while all those series converge, they all converge to different numbers.

nth Term Test

If $a_n \not\to 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

How to use this:

- Check $\lim_{n\to\infty} a_n$ where a_n is the inside of your series. If it's anything except zero, the series diverges by the nth Term Test.
- If $\lim_{n\to\infty} a_n = 0$, the nth Term Test doesn't tell us anything.

Examples:

1. Does $\sum_{k=3}^{\infty} \frac{k+3}{2k-4}$ converge or diverge?
   Answer:
   $\lim_{k\to\infty} \frac{k+3}{2k-4} = \frac{1}{2}$ so $\sum_{k=3}^{\infty} \frac{k+3}{2k-4}$ diverges by the nth Term Test.
2. Does $\sum_{i=2}^{\infty} 2\sin(i)$ converge or diverge?
   Answer:
   $\lim_{i\to\infty} 2\sin(i)$ DNE, so $\sum_{i=2}^{\infty} 2\sin(i)$ diverges by the nth Term Test.

Definition and Convergence/Divergence of a Geometric Series

- A geometric series is a series of the form $\sum_{n=1}^{\infty} ar^{n-1}$ where a, r are numbers and $r \neq 0$. r is said to be the ratio of the series.
- If $|r| < 1$, then $\sum_{n=1}^{\infty} ar^{n-1}$ converges to $\frac{a}{1-r}$.
- If $|r| \geq 1$, then $\sum_{n=1}^{\infty} ar^{n-1}$ diverges.

How to use this:

- If you can manipulate a series into the form $\sum_{n=1}^{\infty} ar^{n-1}$, you can determine the convergence of a series by looking at a, r.

Examples:

1. Does $\sum_{k=1}^{\infty} 6(0.5)^{k-1}$ converge or diverge?
   Answer:
   $\sum_{k=1}^{\infty} 6(0.5)^{k-1} = \frac{6}{1-0.5} = \frac{6}{1/2} = 12$ so it converges. (Because $|0.5| < 1$)

2. Find $\displaystyle\sum_{n=2}^{\infty} \frac{7}{(-2)^{n-2}}$.
   Answer:
   $$\sum_{n=2}^{\infty} \frac{8}{(-3)^{n-2}} = \sum_{n=1}^{\infty} \frac{8}{(-3)^{n-1}} = \sum_{n=1}^{\infty} 8\left(-\frac{1}{3}\right)^{n-1} = \frac{8}{1-(-1/3)} = \frac{8}{4/3} = 6$$
   (Because $|-1/3| < 1$)

3. Does $\displaystyle\sum_{i=1}^{\infty} \frac{\cos(\pi i)}{(2/3)^i}$ converge or diverge?
   Answer:
   $$\sum_{i=1}^{\infty} \frac{\cos(\pi i)}{(2/3)^i} = \sum_{i=1}^{\infty} \frac{(-1)^i}{(2/3)(2/3)^{i-1}} = \sum_{i=1}^{\infty} \left(-\frac{3}{2}\right) \frac{(-1)^{i-1}}{(2/3)^{i-1}} = \sum_{i=1}^{\infty} \left(-\frac{3}{2}\right)\left(-\frac{3}{2}\right)^{i-1}$$
   The series diverges because $|-3/2| \geq 1$.

Telescoping Series Test

$$\sum_{i=1}^{\infty}(a_i - a_{i+1}) = \lim_{n\to\infty} \sum_{i=1}^{n} (a_i - a_{i+1}) = \lim_{n\to\infty} a_1 - a_{n+1}$$

How to use this:

If your series is in the above form, or can be manipulated into the above form, you may compute the value of the series by finding $\lim_{n\to\infty} a_1 - a_{n+1}$.

Examples:

1. Find the value of $\displaystyle\sum_{i=1}^{\infty} 2^{-i} - 2^{-i-1}$.
   Answer
   $$\sum_{i=1}^{\infty} 2^{-i} - 2^{-i-1} = \sum_{i=1}^{\infty} \frac{1}{2^i} - \frac{1}{2^{i+1}} = \lim_{n\to\infty} \frac{1}{2^1} + \frac{1}{2^{n+1}} = \frac{1}{2}$$

2. Does $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(n+1)}$ converge or diverge?
   Answer:
   $$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) = \lim_{n\to\infty} \frac{1}{1} - \frac{1}{n+1} = 1, \text{ so } \sum_{n=2}^{\infty} \frac{1}{n(n+1)} \text{ converges.}$$

3. Does $\displaystyle\sum_{k=1}^{\infty} \ln\left(\frac{k+1}{k}\right)$ converge or diverge?
   Answer:
   $$\sum_{k=1}^{\infty} \ln\left(\frac{k+1}{k}\right) = \sum_{k=1}^{\infty} (-\ln(k) + \ln(k+1)) = \lim_{n\to\infty} -\ln(1) + \ln(n+1) = \infty,$$
   so $\displaystyle\sum_{k=1}^{\infty} \ln\left(\frac{k+1}{k}\right)$ diverges.

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p-Series Test

The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$.

The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges if $p \leq 1$.

Examples:

- These series all converge:

$$\sum_{i=2}^{\infty} -\frac{2}{i^{\pi}} \qquad \sum_{k=-1}^{\infty} \frac{1}{(k+5)^2} \qquad \sum_{n=1}^{\infty} n^{-3} \qquad \sum_{k=3}^{\infty} \frac{2}{3n^{1.01}}$$

- These series all diverge:

$$\sum_{i=2}^{\infty} \frac{i^{\pi}}{2} \qquad \sum_{k=-1}^{\infty} \frac{1}{\sqrt{k+5}} \qquad \sum_{n=1}^{\infty} n^{-1/3} \qquad \sum_{k=3}^{\infty} \frac{2}{3n}$$

Integral Test

Let $\sum_{n=1}^{\infty} a_n$ be a series of positive terms and f be a continuous decreasing function such that $f(n) = a_n$.

- If $\int_1^{\infty} f(x)dx$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
- If $\int_1^{\infty} f(x)dx$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

How to use this:

Compute $\int_1^{\infty} f(x)dx = \lim_{b\to\infty} \int_1^{b} f(x)dx$ to determine the convergence/divergence of $\sum_{n=1}^{\infty} f(n)$.

Example:

Does $\sum_{n=0}^{\infty} 2ne^{-n^2}$ converge or diverge?

Answer:

$\int_1^{\infty} 2xe^{-x^2} = \lim_{b\to\infty} \int_1^{b} 2xe^{-x^2} = \lim_{b\to\infty} [-e^{-x^2}]_1^b = \lim_{b\to\infty} -\frac{1}{e^{b^2}} + \frac{1}{e} = \frac{1}{e}$, so $\sum_{n=0}^{\infty} 2ne^{-n^2}$ converges.

Direct Comparison Test

Let $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$ be series such that $0 \leq a_n \leq b_n$.

- If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
- If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

How to use this:

If you can show the terms of a series is smaller than the terms of a series that converges, it also converges. If you can show the terms of a series is larger than the terms of a series that diverges, it also diverges.

Examples:

1. Does $\sum_{k=2}^{\infty} \frac{k+1}{\sqrt{k^4-1}}$ converge or diverge?

   Answer:

   - $0 \leq \frac{k}{\sqrt{k^4}} \leq \frac{k+1}{\sqrt{k^4-1}}$
   - $\sum_{k=1}^{\infty} \frac{k}{\sqrt{k^4}} = \sum_{k=1}^{\infty} \frac{1}{k}$ diverges (by p-series/Harmonic series).

   Thus by DCT, $\sum_{k=2}^{\infty} \frac{k+1}{\sqrt{k^4-1}}$ diverges.

2. Does $\sum_{n=1}^{\infty} \frac{1}{2^n+n^2}$ converge or diverge?

   Answer:

   - $0 \leq \frac{1}{2^n+n^2} \leq \frac{1}{2^n+2^n}$
   - $\sum_{n=1}^{\infty} \frac{1}{2^n+2^n} = \sum_{n=1}^{\infty} \frac{1}{2^{n+1}} = \sum_{n=1}^{\infty} \frac{1}{4}\left(\frac{1}{2}\right)^{n-1}$ converges (by Geometric Series).

   Thus by DCT, $\sum_{n=1}^{\infty} \frac{1}{2^n+n^2}$ converges.

Limit Comparison Test

Let $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$ be series of positive terms.

- If $0 < \lim_{n\to\infty} \frac{a_n}{b_n} < \infty$, then $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$ both converge or both diverge.
- If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
- If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

How to use this:

- To find out if some $\sum_{n=1}^{\infty} a_n$ converges/diverges, choose some $\sum_{n=1}^{\infty} b_n$ that you know converges/diverges. Evaluate $\lim_{n\to\infty} \frac{a_n}{b_n}$.
- If $\lim_{n\to\infty} \frac{a_n}{b_n}$ is a positive finite number, then $\sum_{n=1}^{\infty} a_n$ matches $\sum_{n=1}^{\infty} b_n$.
- If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
- If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.
- Otherwise you need to choose another test or another series to compare to.

Examples:

1. Does $\sum_{k=2}^{\infty} \frac{k+1}{\sqrt{k^4-1}}$ converge or diverge?

   Answer:

   $$\lim_{k\to\infty} \frac{\frac{k+1}{\sqrt{k^4-1}}}{\frac{1}{k}} = \lim_{k\to\infty} \frac{k^2+k}{\sqrt{k^4-1}} = \lim_{k\to\infty} \frac{k^2}{\sqrt{k^4}} = 1$$

   Thus $\sum_{k=2}^{\infty} \frac{k+1}{\sqrt{k^4-1}}$, $\sum_{k=1}^{\infty} \frac{1}{k}$ ~~both converge~~ or both diverge.

2. Does $\sum_{n=1}^{\infty} \frac{1}{2^n+n^2}$ converge or diverge?

   $\lim_{n\to\infty} \frac{\frac{1}{2^n+n^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{2^n+n^2} = 0$ and $\lim_{n\to\infty} \frac{1}{n^2}$ converges (by p-Series Test)

   Thus by LCT, $\sum_{n=1}^{\infty} \frac{1}{2^n+n^2}$ also converges.

Ratio Test

Let $\sum_{n=1}^{\infty} a_n$ be a series with positive terms.

- If $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} < 1$, then $\sum_{n=1}^{\infty} a_n$ converges.
- If $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.

Root Test

Let $\sum_{n=1}^{\infty} a_n$ be a series with nonnegative terms.

- If $\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} (a_n)^{1/n} < 1$, then $\sum_{n=1}^{\infty} a_n$ converges.
- If $\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} (a_n)^{1/n} > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.

Alternating Series Test

The series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converges if all of the following is true:

1. $a_n > 0$
2. $a_n \geq a_{n+1}$
3. $a_n \to 0$