Calculus II Spring 2010 **Test 3 Practice**

Your Name: Mi

Instructor: Steven Clontz

Draw a box around your final answer. You must show all work to receive credit.

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1. Find a formula (recursive or explicit) for $b_n = \left\langle \frac{2}{1}, \frac{4}{4}, \frac{8}{9}, \frac{16}{16}, \frac{32}{25}, \frac{64}{36}, \dots \right\rangle$

$b_1 = \frac{2}{1} = \frac{2^1}{1^2}$

$b_2 = \frac{4}{4} = \frac{2^2}{2^2}$

$b_3 = \frac{8}{9} = \frac{2^3}{3^2}$

$b_4 = \frac{16}{16} = \frac{2^4}{4^2}$

$$\boxed{b_n = \left\langle \frac{2^n}{n^2} \right\rangle_{n=1}^{\infty}}$$

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2. Compute b_7 for the sequence defined in #1.

$$b_7 = \boxed{\frac{128}{49}}$$

$b_7 = \frac{2^7}{7^2} = \frac{128}{49}$

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3. Show that the sequence $a_n = 1 - \frac{\ln n}{e^n}$ converges or diverges.

$$\lim_{n\to\infty} 1 - \frac{\ln n}{e^n} = 1 - \lim_{n\to\infty} \frac{\ln(n)}{e^n} \stackrel{LH}{=} 1 - \lim_{n\to\infty} \frac{1/n}{e^n} = 1 - \cancel{\lim_{n\to\infty} \frac{1}{ne^n}}$$

$$= 1$$

a_n converges

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4. Show that the series $\sum_{k=2}^{\infty} \frac{1}{4+k^2}$ converges or diverges.

LCT: (Compare to $\sum \frac{1}{k^2}$)

- $\lim_{n\to\infty} \frac{\frac{1}{4+k^2}}{\frac{1}{k^2}} = \lim_{n\to\infty} \frac{k^2}{4+k^2} = \lim_{n\to\infty} \frac{k^2}{k^2} = 1$ ← positive + finite

- $\sum \frac{1}{k^2}$ converges (by p-series)

So by LCT $\sum_{k=2}^{\infty} \frac{1}{4+k^2}$ converges too.

OR DCT

- $0 \leq \frac{1}{4+k^2} \leq \frac{1}{k^2}$
- $\sum \frac{1}{k^2}$ converge

So by DCT (the smaller) $\sum \frac{1}{4+k^2}$ also converges.

5. Show that the series $\sum_{i=1}^{\infty} \frac{i}{(i^2+i)(i+2)}$ converges or diverges. If it converges, bonus points will be awarded if you show its value. (HINT: $\frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2}$)

$$\sum_{i=1}^{\infty} \frac{\cancel{i}}{\underbrace{\cancel{(i^2+i)}}_{i+1}(i+2)} = \sum_{i=1}^{\infty} \frac{1}{(i+1)(i+2)} = \sum_{i=1}^{\infty}\left(\frac{1}{i+1} - \frac{1}{i+2}\right)$$

(Telescoping)

$$= \lim_{n\to\infty}\left(\frac{1}{2} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \cdots + \left(\cancel{\frac{1}{n+1}} - \frac{1}{n+2}\right)$$

$$= \lim_{n\to\infty} \frac{1}{2} - \cancel{\frac{1}{n+2}} = \boxed{\frac{1}{2}}$$ Value & Converges

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$\sum \frac{1}{i^2+3i+2}$

- $0 \le \frac{1}{i^2+3i+2} \le \frac{1}{i^2}$
- $\sum \frac{1}{i^2}$ converges

So by DCT, $\sum$ Converges.

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6. Show that the series $\sum_{n=3}^{\infty} \frac{n!}{3^n}$ converges or diverges.

(Ratio Test)

$$\lim_{n\to\infty} \frac{\frac{(n+1)!}{3^{n+1}}}{\frac{n!}{3^n}} = \lim_{n\to\infty} \frac{\cancel{(n+1)!}}{\cancel{3^{n+1}}} \cdot \frac{\cancel{3^n}}{\cancel{n!}} = \lim_{n\to\infty} \frac{n+1}{3} = \infty > 1$$

By Ratio Test, $\sum_{n=3}^{\infty} \frac{n!}{3^n}$ diverges.

(Also, n^{th} Term Test Works)

$\lim_{n\to\infty} \frac{n!}{3^n} \neq 0$ b/c $n! > 3^n$ eventually

So $\sum \frac{n!}{3^n}$ diverges. ✓

$$\left(\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}\right)$$
if $|r|<1$

7. Show that the series $\displaystyle\sum_{j=0}^{\infty} \frac{1}{2(3^j)}$ converges or diverges. If it converges, bonus points will be awarded if you show its value.

(Diverges for $|r| \geq 1$)

$$\sum_{n=0}^{\infty} \frac{1}{2(3^n)} = \sum_{n=1}^{\infty} \frac{1}{2(3^{n-1})} = \sum_{n=1}^{\infty} \frac{1}{2}\frac{1}{3^{n-1}}$$

Value

$$= \sum_{n=1}^{\infty} \frac{1}{2}\left(\frac{1}{3}\right)^{n-1} = \frac{1/2}{1-1/3} = \frac{1/2}{2/3} = \frac{1}{2}\cdot\frac{3}{2} = \boxed{\frac{3}{4}}$$

converges

Root Test

(NOT value!)

$$\lim_{n\to\infty} \left(\frac{1}{2(3^n)}\right)^{1/n} = \lim_{n\to\infty} \frac{1}{\cancel{2^{1/n}}(3^n)^{1/n}} = \frac{1}{3} < 1$$

$$\left(\lim_{n\to\infty} \sqrt[n]{7} = 1\right)$$

So converges by Root Test

8. Show that $\displaystyle\sum_{n=2}^{\infty} \frac{\cancel{\cos(n\pi)}\,(-1)^n}{\ln n}$ converges absolutely, converges conditionally, or diverges.

Check Abs Conv.

$$\sum_{n=2}^{\infty} \left|\frac{\cancel{(-1)^n}}{\ln n}\right| = \sum_{n=1}^{\infty} \frac{1}{\ln n}$$

LCT

$$\cdot \lim_{n\to\infty} \frac{\frac{1}{\ln n}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{\ln n} \overset{LH}{=} \lim_{n\to\infty} \frac{1}{1/n} = \lim_{n\to\infty} n = \infty$$

$\cdot \sum \frac{1}{n}$ diverges.

Thus by LCT, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\ln n}$ diverges too.

Now check $\displaystyle\sum \frac{(-1)^n}{\ln n}$

Use AST on $\displaystyle\sum_{n=2}^{\infty} (-1)^n \frac{1}{\ln n}$

- $\frac{1}{\ln n}$ is positive ✓
- $\frac{1}{\ln(n)} \geq \frac{1}{\ln(n+1)}$ ✓
- $\displaystyle\lim_{n\to\infty} \frac{1}{\ln n} = 0$ ✓

Thus by AST $\displaystyle\sum (-1)^n \frac{1}{\ln n}$ converges conditionally

9. Show that $\sum_{m=2}^{\infty} \frac{m+1}{(m^2+2m+4)^2}$ converges or diverges.

(Integral Test)

$$\int_1^\infty \frac{x+1}{(x^2+2x+4)^2}\,dx$$

$$= \lim_{b\to\infty} \int_1^b \frac{x+1}{(x^2+2x+4)^2}\,dx$$

Let $u = x^2+2x+4$

$du = (2x+2)\,dx$

$\frac{1}{2}du = (x+1)\,dx$

$$= \lim_{b\to\infty} \int_{x=1}^{x=b} \frac{\frac{1}{2}du}{u^2}$$

$$= \lim_{b\to\infty} \left[-\tfrac{1}{2}u^{-1}\right]_{x=1}^{x=b}$$

$$\lim_{b\to\infty} \left[-\tfrac{1}{2}\frac{1}{x^2+2x+4}\right]_1^b$$

$$\lim_{b\to\infty} -\tfrac{1}{2}\left[\cancel{\frac{1}{b^2+2b+4}} - \frac{1}{1+2+4}\right]$$

(NOT VALUE)

$$= \frac{1}{2(7)} = \frac{1}{14} \leftarrow \text{Converges}$$

So $\sum \frac{m+1}{(m^2+2m+4)^2}$ converges by Int Test

LCT

$$\lim_{m\to\infty} \frac{\frac{m+1}{(m^2+2m+4)^2}}{\frac{m}{(m^2)^2}} = \lim_{m\to\infty} \frac{m+1}{(m^2+2m+4)^2} \cdot \frac{m^{\cancel{4}3}}{\cancel{m}1}$$

$$= \lim_{m\to\infty} \frac{m^3(m+1)}{(m^2+2m+4)^2}$$

$$= \lim_{m\to\infty} \cancel{\frac{m^3(m)}{(m^2)^2}} = 1 \quad \leftarrow +, \text{finite}$$

$\sum \frac{m}{m^4} = \sum \frac{1}{m^3}$ converges by p-series

Thus by LCT, $\sum \frac{m+1}{(m^2+2m+4)^2}$ converges too.

Rankings of unbounded Sequences.

For "big enough" integers n, the following inequalities are true:

Smallest

1

$\log_b n \quad (b>1) \quad (Ex: \log_2 n, \ln n)$

$n^p \quad (0<p<1) \quad (Ex: n^{1/\pi}, \sqrt[3]{n})$

n

$n^p \quad (p>1) \quad (Ex: n^e, n^{1.001})$

$a^n \quad (a>1) \quad (Ex: e^n, \pi^n, 7^n)$

$n!$

n^n

Largest

Examples:

$n^{1000} \leq 1000^n$ eventually $(\text{for } n \geq 1000)$

$\frac{1}{n!} \leq \frac{1}{\ln n}$ eventually $(\text{for } n > 1)$

$\frac{n! + 3^n}{n^n - 1} \leq \frac{n^n + n^n}{n^n - \frac{1}{2} n^n} = 4$ eventually